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Using noise and chaos control to control nonchaotic systems

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Here we show that chaos control techniques can be used to stabilize unstable periodic orbits in a nonchaotic system provided additive noise can be utilized (1) to determine the local dynamics of a chosen orbit, and (2) to move the system's trajectory into the neighborhood of the orbit so that control can be initiated. Using these techniques, we demonstrate that the qualitative dynamics of a nonchaotic system can be altered without using large controls or large parameter shifts. Unlike classical control methods, this approach requires no knowledge of the underlying system equations.

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The original chaos control technique developed by Ott, Grebogi, and Yorke (OGY) [1] is based on the fact that there are an infinite number of unstable periodic orbits embedded within a chaotic attractor. With this approach, a chaotic system is stabilized about one of these periodic orbits by making small perturbations to an accessible system parameter such that the system's trajectory is directed toward the attracting stable manifold of the desired unstable periodic orbit. In this manner, OGY control exploits the extreme sensitivity of chaos to initial conditions. This method is a powerful experimental tool because it requires no knowledge of the underlying system equations, i.e., it models the dynamics of a system directly from its time series. The OGY technique and derivatives thereof have been used to control many different chaotic systems [1,2].

Recently, the principles of chaos control have been utilized by a technique called "tracking" [3] to stabilize underlying unstable periodic orbits in nonchaotic systems. Tracking constrains a system's trajectory within a periodic orbit as the system is moved, via large parameter shifts, through various bifurcations into a regime where the orbit is inherently unstable. Tracking can be initiated either by following a stable periodic orbit into its unstable parameter regime, or by first controlling an unstable periodic orbit in the chaotic regime and then following it out of chaos.

Tracking techniques are useful for stabilizing unstable periodic orbits in nonchaotic systems which can tolerate large parameter shifts. However, such methods are inappropriate for experimental systems which cannot be shifted out of their natural parameter regime to access unstable periodic orbits. Here we describe a method based on OGY control, which is suitable for such systems. With this technique, additive noise is utilized (1) to determine the local dynamics of a chosen unstable periodic orbit in a nonchaotic system, and (2) to move the system's trajectory into the neighborhood of the orbit so that OGY control can be initiated [4]. In contrast to classical control methods, this approach requires no knowledge of the underlying system equations.

To investigate the feasibility of using this method to stabilize unstable periodic orbits in a nonchaotic system, we considered the Hénon map with additive noise, as given by the expression

$$x_n = 1.0 - A x_{n-1}^2 + B x_{n-2} + \xi_n , \tag{1}$$

where A is an adjustable parameter, $B=0.3$, and ξ_n is Gaussian white noise with zero mean and standard deviation σ_ξ. Figure 1 shows the bifurcation diagram of the noise-free (i.e., $\sigma_\xi=0.0$) Hénon map as A is varied from 0.2 to 1.2. It can be seen that as A is increased, the Hénon map undergoes the familiar period-doubling route to chaos. In the present study, we kept the Hénon map out of the chaotic regime by setting A to 1.00—for this value of A, the Hénon map exhibits a stable period-4 cycle (Fig. 1).

Figure 2 shows σ_ξ, x_n, and A for a representative 6000-point control trial. This figure illustrates the destabilizing effects of additive noise and the subsequent utility of OGY control. During the first 250 points, it can be seen that for $\sigma_\xi=0.0$ [Fig. 2(a)] and $A=1.00$ [Fig. 2(c)],

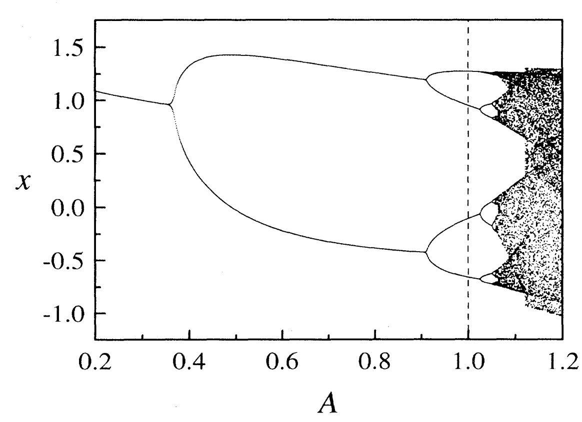

FIG. 1. Bifurcation diagram of the Hénon map [Eq. (1)], with $B=0.3$ and $\sigma_\xi=0.0$ in the parameter range $0.2 \leq A \leq 1.2$. The dashed line at $A=1.00$ identifies the stable period-4 cycle analyzed in this study.

a randomly selected initial condition x_0 quickly settled into the period-4 orbit [Fig. 2(b)] predicted by the bifurcation diagram of Fig. 1. As σ_ξ was uniformly increased over the next 750 points (to a peak value of 0.0707 at $n=1000$), the fluctuations in x_n increased accordingly. (At noise levels below $\sigma_\xi \approx 0.04$, the maximum Lyapunov exponent [5] for the system was negative; at noise levels above $\sigma_\xi \approx 0.04$, the Hénon map was in a state of noise-induced chaos [6], as indicated by a positive maximum Lyapunov exponent.) After the additive noise reached its peak intensity [Fig. 2(a)], a learning stage ($n=1000-2000$) was initiated wherein (1) the position of the unstable period-1 fixed point of the system was estimated, (2) the directions and rates of approach to (stable manifold) and departure from (unstable manifold) the unstable fixed point were linearly approximated, and (3) the sensitivity of the fixed point to small perturbations in parameter A (our chosen control parameter) was determined [7]. After the local dynamics of the unstable period-1 fixed point were determined and once the system wandered into a small neighborhood surrounding the fixed point, OGY control ($n=2000-3500$) was initiated [8]. Adaptive perturbations made to A constrained the system near the period-1 orbit [9] by forcing the system onto the attracting stable manifold of the fixed point whenever the system wandered away from the fixed point. The significant reduction of fluctuations in x_n [Fig. 2(b)] from the precontrol stage to the control stage is evidence of effective OGY control.

The relationship between the additive noise and the effectiveness of OGY control was investigated during a second control stage ($n=4000-5500$), which followed a control inactivation period ($n=3500-4000$). At $n=4250$, a 750-point uniform decrease in σ_ξ was initiated [Fig. 2(a)]. As σ_ξ was decreased, the control of x_n [Fig. 2(b)] improved (i.e., the fluctuations in x_n decreased)

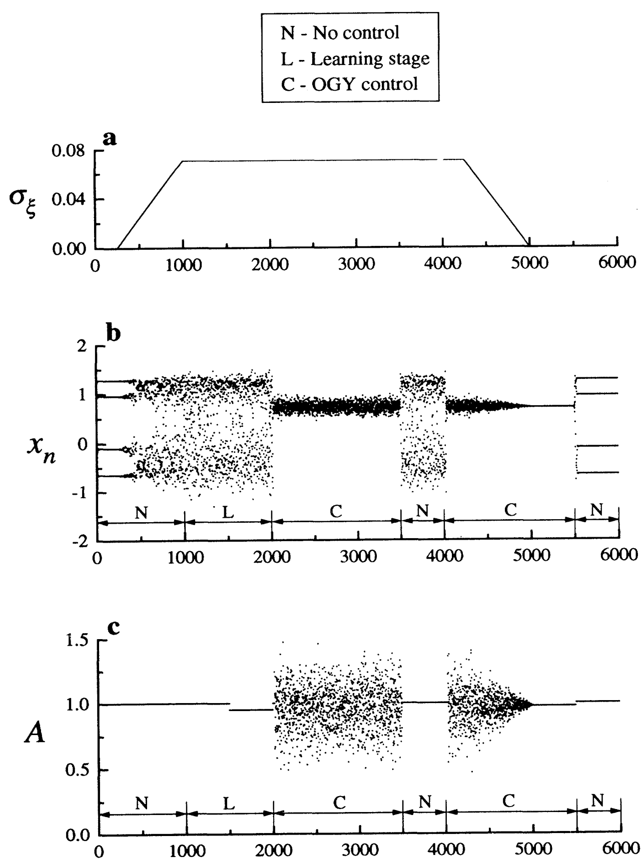

FIG. 2. (a) Standard deviation σ_ξ of the additive noise, (b) output x_n, and (c) system parameter A during a representative 6000-point OGY control trial for the period-4 Hénon map ($A=1.00$) with additive noise. The respective control stages are annotated in (b) and (c).

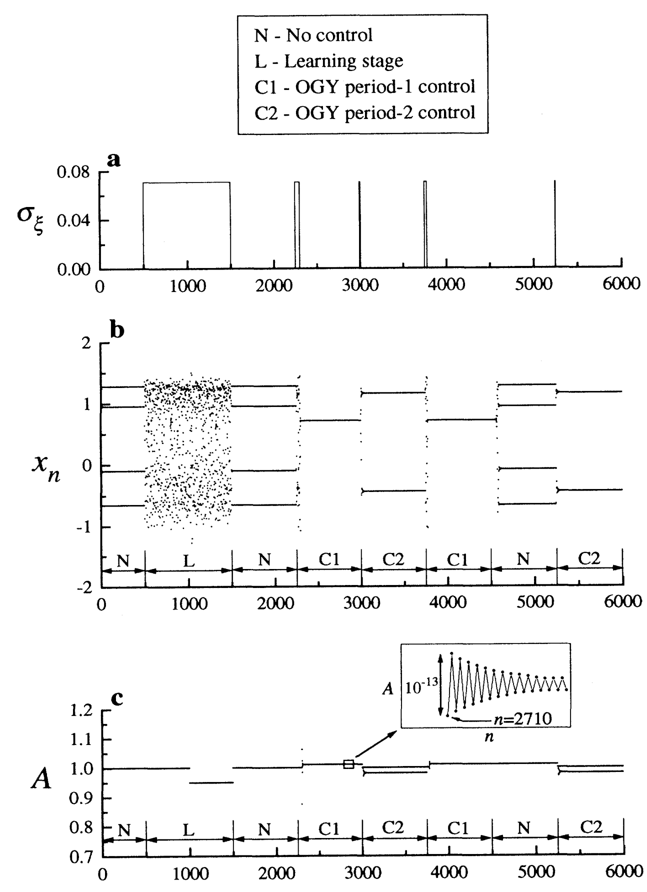

FIG. 3. (a) Standard deviation σ_ξ of the additive noise, (b) output x_n, and (c) system parameter A during a representative 6000-point trial wherein OGY control was used to control the period-4 Hénon map ($A=1.00$) about its underlying unstable period-1 and -2 orbits. The respective control stages are annotated in (b) and (c). The additive noise was used initially to learn the orbit dynamics and then to initiate each control stage. A magnified plot of the variations in A during a portion of an OGY period-1 control stage is shown in the inset in (c).

until a clean period-1 orbit was obtained at $n = 5000$, which corresponded to the noise-free Hénon map, i.e., $\sigma_\xi = 0.0$ [Fig. 2(a)]. Note that because there was no qualitative difference in control effectiveness as the system crossed out of noise-induced chaos (at $\sigma_\xi \approx 0.04$), the success of OGY control in this case can be attributed to the presence of an underlying unstable periodic fixed point in the system, as opposed to the presence of (noise-induced) chaos. In fact, qualitatively similar results could have been obtained by keeping the Hénon map out of a state of noise-induced chaos (i.e., by keeping $\sigma_\xi < 0.04$) throughout the trial shown in Fig. 2—the larger levels of noise served only to decrease the amount of time needed to learn the local dynamics of the unstable period-1 fixed point and subsequently move the system into a small neighborhood surrounding the fixed point (so as to initiate OGY control).

Finally, at $n = 5500$, OGY control was turned off, and parameter A was returned to its original value of 1.00 [Fig. 2(c)]. As expected, x_n quickly departed from the unstable periodic fixed point and returned to its original, stable period-4 orbit [Fig. 2(b)]. Because the estimation of the position of the unstable period-1 fixed point for the period-4 Hénon map was inexact (due to the presence of the additive noise ξ_n), the OGY perturbations to A were made about a value of ~ 0.98 rather than 1.00; hence the jump in A at $n = 5500$ [Fig. 2(c)].

The successful noise-free period-1 control in Fig. 2 ($n = 5000-5500$) suggests that it is possible to use our method to remove higher-order periodicities from the steady-state output of nonchaotic systems. This issue is explored in Fig. 3. The first 500 points of Fig. 3(b) show the output of the noise-free, period-4 Hénon map, i.e., $\sigma_\xi = 0.0$ [Fig. 3(a)] and $A = 1.00$ [Fig. 3(c)]. To determine the local dynamics of the system's underlying unstable period-1 fixed point, noise was added temporarily ($n = 500-1500$) to the system [Fig. 3(a)]. Once the learning stage was completed the noise was turned off [10], and the system quickly returned to its stable period-4 orbit. To initiate period-1 OGY control, a burst of noise [seen as a pulse in Fig. 3(a)] was added to the system at $n = 2250$. Once the system's state point wandered near the unstable period-1 fixed point, the noise was turned off and control was initiated. The system was then successfully controlled about its unstable period-1 orbit by introducing small, adaptive perturbations to parameter A. The inset in Fig. 3(c) shows that even as the system's state point was controlled near the unstable period-1 fixed point, small perturbations in A were always required.

In Fig. 3 (starting at $n = 3000$), we also show that a similar control procedure, which applies OGY-type control interventions every other map iterate [seen as $A \neq 1.00$ for every other point in Fig. 3(c)], can be used to stabilize the system about an unstable period-2 orbit. The position and local dynamics of the unstable period-2 fixed point were determined during the initial learning stage ($n = 500-1500$), along with the position and local dynamics of the unstable period-1 fixed point. Using brief bursts of noise to initiate each additional control stage [Fig. 3(a)], the system was moved successfully from a controlled period-1 orbit to a controlled period-2 orbit ($n = 3000-3750$), and then back to the period-1 orbit ($n = 3750-4500$). To illustrate the instability of the controlled periodic orbits, parameter A was held equal to its final control value ($A \approx 1.01$) after period-1 OGY control was turned off at $n = 4500$ [Fig. 3(c)]. As expected, x_n quickly wandered away from the unstable period-1 orbit and settled into another stable period-4 orbit [Fig. 3(b)]. The final segment of Fig. 3 ($n = 5250-6000$) shows successful period-2 OGY control activated directly from the stable period-4 orbit.

The system controlled in Fig. 3 existed in a state relatively close to the chaotic regime (see Fig. 1). To explore the feasibility of using our technique to stabilize unstable periodic orbits in systems not on the verge of chaos, we considered the period-2 Hénon map with $A = 0.37$ and 0.50, respectively. For $A = 0.37$, the noise intensity needed to implement effective period-1 control [11] was less than that required for the system in Fig. 3, whereas the noise intensity needed for $A = 0.50$ was greater than that required for the system in Fig. 3. For a given system, the noise intensity required to access its unstable periodic orbits will be dependent upon certain system characteristics, such as orbit stability and location. These analyses demonstrate that our method is applicable to systems both near and far from chaos.

This work clearly shows that the small, adaptive perturbations of chaos control can be used, without large parameter shifts, to stabilize unstable periodic orbits in a nonchaotic system. This is in contrast to tracking [3], which requires large parameter shifts to access and stabilize unstable periodic orbits in nonchaotic systems. These findings challenge the previously accepted notion that the qualitative dynamics of a nonchaotic system cannot be altered without large controls and/or large system modifications [1,12]. Importantly, as with OGY control of chaotic systems and in contrast to classical control methods, noise-initiated OGY control of nonchaotic systems requires no knowledge of the underlying system equations. These developments open up a number of potential applications for chaos control techniques. For instance, our method may offer an efficient means for removing higher-order periodicities from the output of nonchaotic, experimental systems. From a physiological standpoint, this could be important given that a number of pathological conditions are associated with the appearance of unwanted higher-order oscillations [13].

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[4] For chaotic systems, this is typically accomplished using a method known as "targeting," which exploits the exponential growth of small parameter perturbations in a chaotic system so as to steer the system's trajectory into the neighborhood of a desired unstable periodic orbit [T. Shinbrot, E. Ott, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **65**, 3215 (1990); T. Shinbrot, C. Grebogi, E. Ott, and J. A. Yorke, Phys. Lett. A **169**, 349 (1992); T. Shinbrot, W. Ditto, C. Grebogi, E. Ott, M. Spano, and J. A. Yorke, Phys. Rev. Lett. **68**, 2863 (1992)]. Such a method is not appropriate for stable, nonchaotic systems, however, given that small disturbances away from stable orbits in such systems eventually decay, i.e., the systems return to their stable orbits.

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[7] The sensitivity of the fixed point to small perturbations in parameter A was determined by reducing A by 5% during the second half of the learning stage.

[8] Following the learning stage, OGY control was activated (i.e., parameter A was first perturbed) when the system's trajectory first wandered into the neighborhood surrounding the unstable periodic fixed point. This occurred at $n=2019$ for the trial shown in Fig. 2.

[9] The fluctuations in x_n during the control stage were due largely to the additive noise ξ_n which caused the system to jump randomly away from the unstable period-1 fixed point.

[10] For an experimental system (with stationary dynamics), this noise-assisted learning stage would need to be completed only once, after which OGY control could be applied repeatedly based on the determined local dynamics of the system's underlying unstable periodic orbits.

[11] The determined noise intensities were those which consistently (over ten trials) produced period-1 stabilization in an amount of time (i.e., number of iterations) similar to that required for period-1 stabilization of the Hénon map with $A=1.00$ and $\sigma_\xi=0.0707$.

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